

REMARKS ON TRIGONOMETRIC FUNCTIONS AFTER EISENSTEIN

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ABSTRACT. We modify the Whittaker-Watson account of the Eisenstein approach to the trigonometric functions, basing these functions independently on the Eisenstein function ε_2 .

0. INTRODUCTION

Eisenstein [E] initiated a novel approach to the theory of the trigonometric functions, based on the meromorphic functions defined by

$$\varepsilon_k(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k}$$

for k a positive integer and $z \in \mathbb{C} \setminus \mathbb{Z}$. These functions were named in honour of Eisenstein by Weil, who elaborated details of the somewhat mystical calculations and further developed the theory in [W]. Of course, this novel approach to the trigonometric functions was but an offshoot or a shadow of the larger theory of elliptic functions. In their account of the Weierstrassian elliptic function theory, Whittaker and Watson [WW] include a very brief introduction to this trigonometric theory by way of illustration.

A little more explicitly, the approach of [E] as explicated in [W] develops the theory of trigonometric functions from the fundamental formula

$$\varepsilon_1(z) = \pi \cot \pi z.$$

This formula is intended as a *definition* of the cotangent function in terms of the positive constant π *defined* by

$$\pi^2 = 6 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

For the identification of $\varepsilon_1(z)$ with $\pi \cot \pi z$ as it is ordinarily understood, we refer to Remmert [R]; this reference also contains an outline of the Eisenstein approach and places it in historical context.

Our purpose here is to modify the approach adopted in [WW] so as to develop the trigonometric functions from the Eisenstein series ε_2 . The approach in [WW] does not lend itself directly to a wholly independent construction of the trigonometric functions, as it incorporates π with its ordinary meaning and makes use of the classical formulae

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

When the approach in [WW] is reformulated so as not to assume π with its ordinary meaning, the proof given there requires independent knowledge of the identity

$$2 \left[\sum_{n=1}^{\infty} \frac{1}{n^2} \right]^2 = 5 \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Our modification circumvents the need for this independent knowledge and indeed has this identity as a consequence. The approach in [WW] essentially identifies $\varepsilon_2(z)$ as $\pi^2 \operatorname{cosec}^2 \pi z$ by virtue of its satisfying certain nonlinear differential equations of first and second order. Our

modification goes beyond this: the reciprocal of ε_2 satisfies the second-order linear differential equation

$$g'' + \left(24 \sum_{n=1}^{\infty} \frac{1}{n^2}\right) g = 2$$

from which the elementary trigonometric functions are immediately in evidence. Our approach has other benefits: for example, it eliminates the need for such tools as the Herglotz trick and the maximum modulus principle, which feature in some accounts of the theory.

1. A MODIFIED APPROACH

Our starting point is the second Eisenstein series, which we rename f for simplicity:

$$f(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

for $z \in \mathbb{C} \setminus \mathbb{Z}$. The indicated series is normally convergent: let $K \subseteq \mathbb{C} \setminus \mathbb{Z}$ be compact and choose $R > 0$ so that K lies in the disc $D_R(0)$; if $z \in K$ and $|n| > R$ then $|z-n| > |n| - R$ so that

$$\sum_{|n|>R} \frac{1}{|z-n|^2} \leq \sum_{|n|>R} \frac{1}{(|n|-R)^2}$$

and the uniformly majorizing series on the right converges by the limit comparison test. As a consequence, $f : \mathbb{C} \setminus \mathbb{Z} \rightarrow \mathbb{C}$ is holomorphic; moreover, f is plainly even and of period one. At each integer, f has a double pole: around zero,

$$f(z) = z^{-2} + \sum_{0 \neq n \in \mathbb{Z}} (z-n)^{-2}$$

where the second summand on the right is holomorphic in the open unit disc, there having Taylor expansion

$$\sum_{0 \neq n \in \mathbb{Z}} (z-n)^{-2} = \sum_{d=0}^{\infty} a_d z^{2d}$$

with

$$a_d = 2(2d+1) \sum_{n=1}^{\infty} n^{-(2d+2)}$$

as follows from the derived geometric series.

We now employ a familiar device, combining suitable derivatives and powers of f so as to eliminate the poles. The Laurent expansion of $f(z)$ about the origin reads

$$f(z) = z^{-2} + a_0 + a_1 z^2 + \dots$$

so that

$$f'(z) = -2z^{-3} + 2a_1 z + \dots$$

and

$$f''(z) = 6z^{-4} + 2a_1 + \dots$$

while

$$f(z)^2 = z^{-4} + 2a_0 z^{-2} + (a_0^2 + 2a_1) + \dots$$

The combination $f'' - 6f^2 + 12a_0 f$ is of course holomorphic in $\mathbb{C} \setminus \mathbb{Z}$ and has period one; its singularities at the integers are removable, in view of the expansion

$$f''(z) - 6f(z)^2 + 12a_0 f(z) = (6a_0^2 - 10a_1) + \dots$$

about the origin, where the ellipsis indicates a power series involving terms of degree two or greater. Removing these singularities, $f'' - 6f^2 + 12a_0 f$ becomes an entire function.

To proceed further, we examine the behaviour of f in the vertical strip

$$S = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1\}.$$

Theorem 1. $f(z) \rightarrow 0$ as $z \rightarrow \infty$ in the strip S .

Proof. Let $z = x + iy \in S$ so that $|x| \leq 1$ and if $n \in \mathbb{Z}$ then $|z - n|^2 = (n - x)^2 + y^2$. If $|n| \leq 1$ then $|z - n|^2 \geq y^2$ while if $|n| > 1$ then $|z - n|^2 \geq (|n| - 1)^2 + y^2$. Accordingly, it follows that

$$|f(z)| \leq \frac{3}{y^2} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2 + y^2}.$$

As $z \rightarrow \infty$ in S we need only inspect the second summand on the right. For any N we have

$$\sum_{n=1}^{\infty} (n^2 + y^2)^{-1} = \sum_{1 \leq n \leq N} (n^2 + y^2)^{-1} + \sum_{n > N} (n^2 + y^2)^{-1}.$$

Let $\varepsilon > 0$: choose N so that $\sum_{n > N} n^{-2} < \varepsilon$; it follows that if $|y| > \sqrt{N/\varepsilon}$ then

$$\sum_{n=1}^{\infty} (n^2 + y^2)^{-1} \leq Ny^{-2} + \sum_{n > N} n^{-2} < 2\varepsilon.$$

□

We now see that the entire function $f'' - 6f^2 + 12a_0f$ is as trivial as can be.

Theorem 2. *The meromorphic function f satisfies*

$$f'' - 6f^2 + 12a_0f = 0.$$

Proof. The argument of Theorem 1 adapts easily to show that the second derivative $f''(z) = 6 \sum_{n \in \mathbb{Z}} (z - n)^{-4}$ also tends to 0 as $z \rightarrow \infty$ in S . The entire function $f'' - 6f^2 + 12a_0f$ is thus bounded in S and so bounded on \mathbb{C} by periodicity. According to the Liouville theorem, $f'' - 6f^2 + 12a_0f$ is constant; the value of this constant is 0 because $f'' - 6f^2 + 12a_0f$ vanishes at infinity. □

Thus the constant term $6a_0^2 - 10a_1$ in the expansion of $f'' - 6f^2 + 12a_0f$ about the origin is zero. When we substitute the expressions for a_0 and a_1 and then simplify, we obtain the identity

$$2 \left[\sum_{n=1}^{\infty} \frac{1}{n^2} \right]^2 = 5 \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Theorem 3. *The meromorphic function f satisfies*

$$(f')^2 - 4f^3 + 12a_0f^2 = 0.$$

Proof. Multiply the equation of Theorem 2 by $2f'$ to obtain

$$2f'f'' - 12f'f^2 + 24a_0f'f = 0$$

and then integrate to obtain

$$(f')^2 - 4f^3 + 12a_0f^2 = c$$

for some $c \in \mathbb{C}$. As f and (similarly) f' vanish at infinity, $c = 0$. □

Theorem 4. *The function f is nowhere zero.*

Proof. Theorem 2 and Theorem 3 tell us that if we write $2p(w) = 4w^3 - 12a_0w^2$ then $f'' = p' \circ f$ and $(f')^2 = 2p \circ f$. An elementary induction shows that each even-order derivative of f is a polynomial in f with vanishing constant term: for the inductive step, if $f^{(2d)} = q \circ f$ then $f^{(2d+1)} = (q' \circ f)f'$ and $f^{(2d+2)} = (2q''p + q'p') \circ f$; the square of each odd-order derivative of f is then also a polynomial in f with vanishing constant term. Finally, if f were to vanish at $a \in \mathbb{C} \setminus \mathbb{Z}$ then all its derivatives would vanish at a ; the Identity Theorem would then force f itself to vanish, which is absurd. □

We may now introduce the reciprocal function $g = 1/f$: as f is a nowhere-zero meromorphic function with a double pole at each integer, g is an entire function with a double zero at each integer; as f is even and of period one, g is even and of period one.

Theorem 5. *The entire function g satisfies*

$$g'' + 12a_0g = 2.$$

Proof. Simply differentiate and then substitute from Theorem 2 and Theorem 3: $g' = -f^{-2}f'$ so that $g'' = 2f^{-3}(f')^2 - f^{-2}f'' = 2f^{-3}(4f^3 - 12a_0f^2) - f^{-2}(6f^2 - 12a_0f) = 2 - 12a_0g$ as required. \square

Recall that g has a double zero at the origin; accordingly, the second-order differential equation displayed in Theorem 5 is supplemented by the initial data $g(0) = 0$ and $g'(0) = 0$.

At this point, it is quite clear that our approach has made contact with the elementary trigonometric functions. Define the positive number π by

$$\pi^2 := 3a_0 = 6 \sum_{n=1}^{\infty} n^{-2}.$$

Define the function $c : \mathbb{C} \rightarrow \mathbb{C}$ by the rule that if $z \in \mathbb{C}$ then

$$c(z) := 1 - 2\pi^2 g(z/2\pi).$$

The entire function c has period 2π ; this inbuilt periodicity is a special feature of the Eisenstein approach. Further, a direct calculation reveals that it satisfies the initial value problem

$$c'' + c = 0; \quad c(0) = 1, \quad c'(0) = 0.$$

As an entire function, its Taylor series about the origin is consequently

$$c(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Thus c is precisely the cosine function, from which flows the whole theory of trigonometric functions. Incidentally, a duplication formula for the cosine function shows that $f(z) = \pi^2 \operatorname{cosec}^2 \pi z$.

We close by remarking on ways in which our approach varies from the approach in [WW]. First of all, [WW] incorporates π in the theory from the very start; its removal from the function there analyzed yields f . Our Theorem 1 improves the [WW] observation that $f(z)$ is *bounded* as $z \rightarrow \infty$ in the strip $\{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1/2\}$; the weaker result means that [WW] must assume the identity $2[\sum_{n=1}^{\infty} n^{-2}]^2 = 5 \sum_{n=1}^{\infty} n^{-4}$ in order to conclude that the bounded function $f'' - 6f^2 + 12a_0f$ is identically zero. Our Theorem 4 to the effect that f never vanishes permits us to pass directly to its reciprocal g and thence to the elementary second-order linear differential equation in Theorem 5; by contrast, [WW] essentially stops short at the nonlinear differential equations that we display in Theorem 2 and Theorem 3.

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